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SUPERPOSITION LAWS FOR NONLINEAR EQUATIONS ARISING IN OPTIMAL CONTROL THEORY

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SUPERPOSITION LAWS FOR NONLINEAR EQUATIONS ARISING IN OPTIMAL CONTROL THEORY

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RESUME

Lois de superposition pour des équations non linéaires utilisées dans la théorie du contrôle optimal.

Nous donnons les formules de représentation de la solution générale de l'équation de Riccati matricielle

$$\dot{W} = A+WB+CW+WDW \qquad (W \in R^{N\times N})$$

Les "n-représentations" impliquent n solutions particulières avec n=1,...,5. La 5-représentation est une "loi de superposition" : elle exprime la solution générale explicitement en fonction de 5 solutions connues et de N^2 constantes arbitraires (pour $N\geq 2$). Les formules de représentation peuvent être utilisées pour les calculs numériques. Les représentations 4 et 5 sont particulièrement utiles lorsqu'une solution W(t) a une singularité pour une valeur $t=t_0$. Elles clarifient également les propriétés de l'espace de solutions : les éléments de la matrice W sont des fonctions méromorphes de t n'ayant que des pôles simples. Nous discutons la relation entre les formules de représentation et les résultats déjà connus.

ABSTRACT

Representation formulas are given for the general solution of the N×N matrix Riccati equation

$$\dot{W} = A+WB+WC+WDW$$

using n known solutions, with n=1,...,5 (n-representations). The 5-representations is a "superposition formula", in that it expresses the general solution explicitly as a function of 5 particular solutions and N^2 arbitrary constants ($N\geq 2$), using no further information. The representation formulas can be used in numerical calculations. The 4- and 5-representations are specially useful when a solution W(t) has a singularity for some finite $t=t_0$. They also clarify the properties of the solution space : the matrix elements of W(t) are meromorphic functions of t having simple poles as the only possible singularities. The relation between the representation formulas and previously known results is discussed.

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PAPIER RECUPÉRÉ ET RECYCLÉ

1. INTRODUCTION

For the purposes of this paper, we shall call a "superposition law" a formula expressing the general solution of a system of ordinary differential equations in terms of a finite number of particular solutions and a sufficient number of arbitrary constants. With this definition a system of homogeneous linear differential equations becomes a special case: the superposition law presents the general solution as a linear combination of n particular solutions.

A series of recent publications has been devoted to extending these essentially linear techniques to certain classes of nonlinear equations [1,2,3].

S.Lie has provided a criterion for the existence of a superposition law, or in his terminology, a fundamental set of solutions, for a system of n first order ordinary differential equations [4]. Namely, he considers the system

$$\frac{dy^\mu}{dt} = \eta^\mu(y^1,\dots,y^n,t), \qquad 1 \le \mu \le n, \tag{1}$$

where η^μ are given functions. It allows a superposition law

$$\underset{\sim}{y} = \underset{\sim}{F}(\underset{\sim}{y}_{(1)},\dots,\underset{\sim}{y}_{(m)}, c_1,\dots,c_n), \tag{2}$$

where $\underset{\sim}{y}_{(1)},\dots,\underset{\sim}{y}_{(m)}$ are particular solutions of (1) and $c_1,\dots,c_n$ are constants, if and only if (1) can be rewritten as

$$\frac{d\underset{\sim}{y}}{dt} = Z_1(t)\underset{\sim}{\xi}_1(\underset{\sim}{y}) + \dots + Z_r(t)\underset{\sim}{\xi}_r(\underset{\sim}{y}) \tag{3}$$

and the vector fields

$$X_a = \sum_{\mu=1}^{n} \xi_a^\mu(\underset{\sim}{y}) \frac{\partial}{\partial y^\mu}, \tag{4}$$

defined by the coefficients $\xi_a^\mu(y)$ in (3) generate a finite dimensional Lie algebra:

$$[X_a, X_b] = f_{ab}^c X_c, \qquad 1 \le a,b,c \le \ell; \quad r \le \ell \tag{5}$$

(f_{ab}^c are the structure constants).

In particular, the matrix Riccati equation (MRE)

$$\frac{dW}{dt} \equiv \dot{W} = A + WB + CW + WDW, \qquad W(s) = W_0, \tag{6}$$
$$W \in \mathbb{R}^{N\times K}, \quad A \in \mathbb{R}^{N\times K}, \quad B \in \mathbb{R}^{K\times K}, \quad C \in \mathbb{R}^{N\times N}, \quad D \in \mathbb{R}^{K\times N},$$

where A,B,C and D are given matrices, in general depending on the independent variable t, has been shown to satisfy Lie's criterion [3]. For square matrix Riccati equations $N = K$ the general solution was expressed in terms of five particular solutions (for arbitrary $N \ge 2$). In the case of the symplectic MRE, when $A = A^T$, $D = D^T$, $C = B^T$ and $W_0 = W_0^T$, four particular solutions suffice [3] (the superscript T denotes transposition).

The vector fields (4) associated with the MRE (6) in general span the Lie algebra $sl(N+K,\mathbb{R})$; for $N=K$, $A = A^T$, $D = D^T$ and $C = B^T$, they span the symplectic algebra $sp(2N,\mathbb{R})$ [3].

The purpose of this article is to present explicit superposition formulas for the matrix Riccati equation that are convenient for numerical calculations. The formulas are an improvement over those considered previously [3] and in particular we derive a global superposition formula for the symplectic matrix Riccati equation (valid for all $t \ge 0$). We also relate the results on superposition laws to previously known results used in optimal control theory [5,...,14].

2. REPRESENTATION FORMULAS

Let us consider the square matrix Riccati equation (6) and assume that A,B,C and D are continuous functions of t with values in $\mathbb{R}^{N\times N}$. Equation (6) then has, at least locally, a continuously differentiable solution W and the following system of linear equations also has a solution (X,Y):

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} C & A \\ -D & -B \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad X(s) = W(s);\ Y(s) = I\ . \tag{7}$$

Furthermore, for small $|t-s|$ we have $\det Y(t) \neq 0$ and the solution of (6) is given by

$$W = XY^{-1}\ . \tag{8}$$

We shall now give explicit formulas for W using some already known solutions of the MRE (6). A formula using n particular solutions will be called an n-representation of W. In this section we treat n=1,2, and 3.

1-representation

Let W_1 be a solution of (6), for some initial condition $W_1(s)$, and S be a solution of the linear inhomogeneous equation

$$\dot{S} = -S\tilde{C} - \tilde{B}S - D \tag{9}$$

$$\tilde{B} = B+DW_1, \qquad \tilde{C} = C+W_1D\ . \tag{10}$$

We define the transformation

$$\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} -S & I+SW_1 \\ I & -W_1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} W_1 & I+W_1S \\ I & S \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \tag{11}$$

which diagonalizes equations (7):

$$\begin{pmatrix} \dot{X}_1 \\ \dot{Y}_1 \end{pmatrix} = \begin{pmatrix} -\tilde{B} & 0 \\ 0 & \tilde{C} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \quad X_1(s)=I+S(s)(W_1(s)-W(s)), \; Y_1(s)=W(s)-W_1(s). \tag{12}$$

Defining $G(\tilde{B},t,s)$ and $H(\tilde{C},t,s)$ as (local) solutions of the linear homogeneous equations

$$\begin{cases} \frac{d}{dt} G(\tilde{B},t,s) = -\tilde{B}(t)G(\tilde{B},t,s), & G(\tilde{B},s,s) = I, \\ \frac{d}{dt} H(\tilde{C},t,s) = -H(\tilde{C},t,s)\tilde{C}(t), & H(\tilde{C},s,s) = I, \end{cases} \tag{13}$$

we have

$$X_1(t) = G(\tilde{B},t,s)X_1(s), \quad Y_1(t) = H^{-1}(\tilde{C},t,s)Y_1(s) \tag{14}$$

and at least for small $|t-s|$:

$$G^{-1}(\tilde{B},t,s) = H(-\tilde{B},t,s) \; . \tag{15}$$

The quantity

$$\Delta_1(t) = W(t)-W_1(t) = Y_1(t)Y^{-1}(t)$$

is now easily seen to be given by

$$\Delta_1(t) = H^{-1}(\tilde{C},t,s)\Delta_1(s)\{G(\tilde{B},t,s)[I-S(s)\Delta_1(s)]+S(t)H^{-1}(\tilde{C},t,s)\Delta_1(s)\}^{-1}, \tag{16}$$

where $S(s)$ can be chosen arbitrarily. Take $S(s) = 0$ and call $S_0(t)$ the corresponding solution of (9). We obtain the so-called "addition formula"

$$\Delta_1(t) = H^{-1}(\tilde{C},t,s)\Delta_1(s)[G(\tilde{B},t,s)+S_0(t)H^{-1}(\tilde{C},t,s)\Delta_1(s)]^{-1} \; . \tag{17}$$

Formula (17) is well known in the literature on the MRE and two point boundary value problems [5,6]. It is also at the origin of the so-called "integration-free algorithms" [7] and of some efficient use of the Chandrasekhar equations [8].

Notice that formula (17) makes use of only 1 solution of the MRE, but does involve 3 solutions of the linear differential equations (9) and (13). Notice also that the expression is equally valid for rectangular MRE, indeed the equality N=K was never used [12].

In the special case when the MRE has constant coefficients and W_1 can be chosen to be a constant solution, equations (13) can be integrated explicitly:

$$G(\tilde{B},t,s) = e^{-\tilde{B}(t-s)}, \quad H(\tilde{C},t,s) = e^{-\tilde{C}(t-s)} . \tag{18}$$

Formula (17) then reduces to

$$\Delta_1(t) = e^{\tilde{C}(t-s)}\Delta_1(s)[e^{-\tilde{B}(t-s)}+S_0(t)e^{\tilde{C}(t-s)}\Delta_1(s)]^{-1}.$$

2-representation

Let $W_1(t)$ and $W_2(t)$ be two solutions of the MRE (6) and assume

$$\det(W(s)-W_1(s)) \neq 0, \quad \det(W_2(s)-W_1(s)) \neq 0 . \tag{19}$$

Since the rank of the difference between any two solutions of the MRE is a constant in each interval where the difference is defined [5], we have for $|t-s|$ small

$$\det(W(t)-W_1(t)) \neq 0, \quad \det(W_2(t)-W_1(t)) \neq 0 . \tag{20}$$

Let S_1 and S_2 be solutions of equation (9) with initial values $S_1(s) = [W(s)-W_1(s)]^{-1}$, $S_2(s) = [W_2(s)-W_1(s)]^{-1}$. For t near s we then have

$$S_1(t) = [W(t)-W_1(t)]^{-1}, \quad S_2(t) = [W_2(t)-W_1(t)]^{-1} . \tag{21}$$

Set $T_{12}(W) = S_1(t)-S_2(t)$. We have

$$T_{12}(W) = (W-W_1)^{-1}(W_2-W)(W_2-W_1)^{-1} = (W_2-W_1)^{-1}(W_2-W)(W-W_1)^{-1} \tag{22}$$

and

$$\dot{T}_{12}(W) = -T_{12}(W)\tilde{C}-\tilde{B}T_{12}(W), \tag{23}$$

so that

$$T_{12}(W)(t) = G(\tilde{B},t,s)T_{12}(W)(s)H(\tilde{C},t,s) . \tag{24}$$

Let us introduce the following linear and nonlinear operators on the algebra of square matrices $\mathbb{R}^{N\times N}$:

$\mathcal{W}(t,s)$ is the Riccati (nonlinear) evolution operator defined as:

$$\mathcal{W}(t,s):W(s) \to W(t) \quad \text{where } W \text{ is a solution of (6)} \tag{25}$$

$L(\tilde{B},\tilde{C},t,s)$ is a linear evolution operator defined as:

$$L(\tilde{B},\tilde{C},t,s): \mathcal{W} \in \mathbb{R}^{N\times N} \to G(\tilde{B},t,s)\mathcal{W}H(\tilde{C},t,s) . \tag{26}$$

We have the usual composition laws for these operators:

$$\begin{cases} \mathcal{W}(t,\sigma)\circ \mathcal{W}(\sigma,s) = \mathcal{W}(t,s), \quad \mathcal{W}(s,s) = I, \\ L(\tilde{B},\tilde{C},t,\sigma)\circ L(\tilde{B},\tilde{C},\sigma,s) = L(\tilde{B},\tilde{C},t,s), \quad L(\tilde{B},\tilde{C},s,s) = I \end{cases} \tag{27}$$

$T_{12}(t)$ is a one parameter family of homographic transformations:

$$T_{12}(t): \mathcal{W} \in \mathbb{R}^{N\times N} \to (\mathcal{W}-W_1(t))^{-1}(W_2(t)-\mathcal{W})(W_2(t)-W_1(t))^{-1} \tag{28}$$

defined on the set of $\mathcal{W}$'s such that $\det(\mathcal{W}-W_1(t)) \neq 0$. We remark that the inverse homographic transformation is well defined on the range of $T_{12}(t)$ because $T_{12}(t)(\mathcal{W}) = T_{12}(t)(\mathcal{W}')$ implies $\mathcal{W} = \mathcal{W}'$ and furthermore

$$T_{12}^{-1}(t): \mathcal{W} \in \mathbb{R}^{N\times N} \to [W_2(t)+W_1(t)\mathcal{W}(W_2(t)-W_1(t))][I+\mathcal{W}(W_2(t)-W_1(t))]^{-1} \tag{29}$$

We can now rewrite (24) in a following form which makes the time dependance of W(t) more transparent and formulate the result as a theorem.

<u>THEOREM 1</u>. Let W_1, W_2 be two solutions of (6) such that $\det(W_2(s)-W_1(s)) \neq 0$, then we have, at least for small $|t-s|$:

$$\mathcal{W}(t,s) = T_{12}^{-1}(t)\circ L(\tilde{B},\tilde{C},t,s)\circ T_{12}(s) \tag{30}$$

on the set of W(s) such that $\det(W(s)-W_1(s)) \neq 0$. □

Let us consider some particular cases:

- <u>Symplectic MRE</u>. This corresponds to the case

$$C(t) = B^T(t), \quad A(t) = A^T(t), \quad D(t) = D^T(t). \tag{31}$$

We can choose $W_1(s) = W_1(s)^T$, so that $W_1(t) = W_1(t)^T$. We then obtain:

$$\tilde{C}^T = \tilde{B}, \qquad H(\tilde{C},t,s) = G(\tilde{B},t,s)^T . \tag{32}$$

- MRE with constant coefficients

$$A(t) \equiv A, \quad B(t) \equiv B, \quad C(t) \equiv C, \quad D(t) \equiv D \tag{33}$$

If we can choose $W_1(t) \equiv W_1$ (time independent), (24) becomes

$$L(\tilde{B},\tilde{C},t,s)(\mathcal{W}) = e^{-\tilde{B}(t-s)}\mathcal{W}e^{-\tilde{C}(t-s)}$$

which leads to:

$$W(t) = \{W_2(t)+W_1e^{-\tilde{B}(t-s)}T_{12}(W)(s)e^{-\tilde{C}(t-s)}(W_2(t)-W_1)\}\{I+e^{-\tilde{B}(t-s)}T_{12}(W)(s)e^{-\tilde{C}(t-s)} \times(W_2(t)-W_1)\}^{-1} . \tag{34}$$

- Symplectic MRE with constant coefficients

We assume that (31) and (32) hold. In order to be able to choose W_1 and W_2 as time independent solutions of the MRE, we furthermore assume that

$$\begin{cases} A \geq 0, \quad D \leq 0, \\ (B,(-D)^{1/2}) \text{ is controllable}, \quad (A^{1/2},B) \text{ is observable} . \end{cases} \tag{35}$$

Then we know that the algebraic Riccati equation (ARE)

$$A+WB+B^TW+WDW = 0 \tag{36}$$

has a positive definite solution W_1 and a negative definite solution W_2:

$$W_1 > 0, \quad W_2 < 0. \tag{37}$$

Now we can give a global 2-representation result of the type (30):

THEOREM 2. Under assumptions (31), (33), and (35), take W_1 and W_2 as solutions of the ARE (36) satisfying (37) to define T_{12} and $L(\tilde{B},\tilde{C},t,s)$. We have:

$$\begin{cases} \mathcal{W}(t,s) \equiv \mathcal{W}(t-s), \quad T_{12}(t) \equiv T_{12}, \\ L(\tilde{B},\tilde{C},t,s)(W) = e^{\tilde{B}(t-s)}\mathcal{W}e^{\tilde{B}^T(t-s)} \equiv L(t-s)(\mathcal{W}) \end{cases} \tag{38}$$

and $\forall t \geq s$

$$\mathcal{W}(t) = T_{12}^{-1}\circ L(t)\circ T_{12} \tag{39}$$

on the set of $W(s)$ such that $W(s) > W_2$.

PROOF. The only point to verify is the global existence of W with $W(s) > W_2$, which was proven e.g. in Ref. [9]. □

REMARKS. Roughly speaking, Theorem 1 states that the nonlinear Riccati evolution operator is homographically similar to a linear operator. As a first consequence of this theorem we can already claim that there exists a superposition law for the MRE (6) using at most N^2+2 solutions. In fact this would be a superposition law "homographically similar" to a linear one. In the sequel we shall show how five solutions suffice when using more general nonlinear laws.

Formulas of type (39) seem to have been first proven in [10] where furthermore assumptions (35) are weakened and results are shown to be valid for infinite dimensional parabolic systems.

3-representations

We now return to the general case of MRE's with time dependent coefficients.

Let W_3 be a third solution of (6) such that $T_{12}(W_3)$ and $T_{12}(W_3)^{-1}$ are defined for small $|t-s|$. A sufficient condition for this is

$$\det(W_1(s)-W_2(s)) \neq 0, \quad \det(W_2(s)-W_3(s)) \neq 0, \quad \det(W_3(s)-W_1(s)) \neq 0. \tag{40}$$

For all W satisfying $\det(W(t)-W_1(t)) \neq 0$ we define the "matrix anharmonic ratio"

$$T_{123}(W) = T_{12}(W)\cdot T_{12}(W_3)^{-1} = (W-W_1)^{-1}(W_2-W)(W_2-W_3)^{-1}(W_3-W_1). \tag{41}$$

If W is a solution of (6) it is easy to verify that, at least locally, we have:

$$\dot{T}_{123}(W) = -\tilde{B}T_{123}(W)+T_{123}(W)\tilde{B} = -[\tilde{B},T_{123}(W)]. \tag{42}$$

Let us introduce now the following family of homographic transformations associated with $T_{123}(W)$:

$$\mathcal{T}_{123}(t):\mathcal{W} \in R^{n\times n} \to (\mathcal{W}-W_1(t))^{-1}(W_2(t)-\mathcal{W})(W_2(t)-W_3(t))^{-1}(W_3(t)-W_1(t)) \tag{43}$$

where the image of $\mathcal{W}$ is its "matrix anharmonic ratio" with W_1,W_2,W_3.

The inverse of $\mathcal{T}_{123}(t)$ is well defined on the range of $\mathcal{T}_{123}(t)$ and is given by:

$$\mathcal{T}_{123}(t)^{-1}:\mathcal{W} \in \mathbb{R}^{N\times N} \to \{W_2(t)+W_1(t)\mathcal{W}(W_3(t)-W_1(t))^{-1}(W_2(t)-W_3(t))\}$$
$$\times\{I+\mathcal{W}[W_3(t)-W_1(t))^{-1}(W_2(t)-W_3(t)]\}^{-1} . \tag{44}$$

We formulate the above results as a theorem:

<u>THEOREM 3.</u> Let W_1,W_2,W_3 be three solutions of (6) such that (40) holds, then we have, at least for small $|t-s|$

$$\mathcal{W}(t,s) = \mathcal{T}_{123}^{-1}(t)\circ L(\tilde{B},-\tilde{B},t,s)\circ\mathcal{T}_{123}(s) \tag{45}$$

on the set of $W(s)$ such that $\det(W(s)-W_1(s)) \neq 0$. □

REMARKS. From formula (42) we deduce in particular that the anharmonic ratio $T_{123}(W)$ is similar to a constant matrix

$$T_{123}(W)\ (t) = G(\tilde{B},t,s)\ T_{123}(W)\ (s)G^{-1}(\tilde{B},t,s) . \tag{46}$$

This is a known result [11],[5]. The problem of finding superposition laws for the MRE is now reduced to the same problem for the linear equations associated with $L(\tilde{B},\tilde{B}^T,t,s)$ (symplectic case) and $L(\tilde{B},-\tilde{B},t,s)$ (general case). This problem is studied in the following section.

3. <u>4 AND 5- REPRESENTATIONS: SUPERPOSITION LAWS FOR THE MRE</u>

Let us introduce the following notations:

λ_i, $i=0,\dots,N-1$ are real numbers with $\lambda_i \neq \lambda_j$, for $i\neq j$.

$P_n(\lambda)$, $n=0,\ldots,N-1$ are polynomials over $\mathbb{R}$ and $\mathbb{R}^{N\times N}$ such that over $\mathbb{R}$: $P_n(\lambda_i) = \delta_{in}$, $i,n=0,\ldots,N-1$, i.e.

$$P_n(\lambda) = \frac{\prod_{i=0,i\neq n}^{N-1} (\lambda-\lambda_i)}{\prod_{i=0,i\neq n}^{N-1} (\lambda_n-\lambda_i)} \tag{47}$$

Δ is a diagonal matrix, the diagonal being $(\lambda_0,\ldots,\lambda_{N-1})$ and U is an N by N matrix with all entries equal to 1:

$$\Delta_{ij} = \delta_{ij}\cdot\lambda_i, \quad U_{ij} = 1, \quad i,j=0,1,\ldots,N-1 \ . \tag{48}$$

The basic facts leading to the superposition laws are given by the following lemmas:

<u>LEMMA 1.</u> The solutions of eq. (42) form an associative algebra for the usual product of matrices, i.e.: $\forall t,s$, $\forall T_1,T_2 \in \mathbb{R}^{N\times N}$

$$(L(\widetilde{B},-\widetilde{B},t,s)T_1)\cdot(L(\widetilde{B},-\widetilde{B},t,s)T_2) = L(\widetilde{B},-\widetilde{B},t,s)(T_1\cdot T_2) \ . \tag{49}$$

PROOF. The result follows from the identity

$$L(\widetilde{B},-\widetilde{B},t,s)T = G(\widetilde{B},t,s)TG(\widetilde{B},t,s)^{-1} \ . \quad \square \tag{50}$$

We set now, for every matrix $M \in \mathbb{R}^{N\times N}$

$$\mathbb{P}(M) = [P_0(M),\ldots,P_{N-1}(M)], \quad \mathbb{P}^T(M) = (P(M^T))^T, \tag{51}$$

the elements of $\mathbb{R}^{N\times N^2}$ and $\mathbb{R}^{N^2\times N}$ respectively, obtained by arranging the P_i's along a row and along a column.

We shall denote $M \otimes N$ the Kronecker product of matrices.

<u>LEMMA 2.</u> For every $M \in \mathbb{R}^{N\times N}$, we have

$$M = \mathbb{P}(\Delta)\cdot(M\otimes U)\cdot\mathbb{P}^T(\Delta) \ . \tag{52}$$

PROOF. Let $\{E_{mn}, m,n=0,\dots,N-1\}$ be the usual basis of $\mathbb{R}^{N\times N}$ with $(E_{mn})_{ij} = \delta_{mi}\delta_{nj}$. It is easy to see that

$$E_{nn} = P_n(\Delta), \quad E_{mn} = E_{mm}UE_{nn}, \qquad m,n=0,\dots,N-1 \tag{53}$$

so that $M = (M_{mn})$ can be written

$$M = \sum_{m,n=0}^{N-1} M_{mn}P_m(\Delta)UP_n(\Delta)$$

and (52) follows. □

We are now able to give our main results for square MRE.

Let us choose W_3, W_4 solutions of the MRE such that

$$\mathcal{T}_{12}(W_3)(s) = I, \quad \mathcal{T}_{123}(W_4)(s) = \Delta . \tag{54}$$

For example, if we choose

$$W_1(s) = 0, \quad W_2(s) = I, \quad W_3(s) = \frac{1}{2} I \tag{55}$$

we have

$$W(s) = (I+\mathcal{T}_{123}(W)(s))^{-1} \tag{56}$$

as long as the inverse exists. In particular we can choose:

$$W_4(s) = (I+\Delta)^{-1} = \operatorname{diag}\left(\frac{1}{1+\lambda_0},\dots,\frac{1}{1+\lambda_{N-1}}\right), \qquad \lambda_i \neq -1, \ i=0,\dots,N-1. \tag{57}$$

Note that with the choice (55), $\mathcal{T}_{12}$ and $\mathcal{T}_{123}$ now depend also upon s and we shall write $\mathcal{T}_{12}(W)(t,s)$ and $\mathcal{T}_{123}(W)(t,s)$. In the symplectic case we shall also use $X_1(t,s)$, the solution of

$$\frac{\partial}{\partial t} X_1(t,s) = \widetilde{B}(t)X_1(t,s), \qquad X_1(s,s) = \mathbb{1} \tag{58}$$

where $\mathbb{1} \in \mathbb{R}^N$ is the vector $(1,\dots,1)^T$.

Let us now present the superposition formula for the symplectic MRE, i.e. equation (6) with coefficients satisfying (31).

<u>THEOREM 4.</u> Let W_1, W_2, W_3, W_4 be solutions of the symplectic MRE satisfying (40) and (54) (take for example these solutions as in (55) and (57)) and set:

$$\mathcal{P}_{1234}(M)(t,s) = \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s))\cdot[M\otimes(X_1(t,s)\cdot X_1(t,s)^T)]$$
$$\cdot\mathbb{P}(\mathcal{T}_{123}(W_4)(t,s))^T, \qquad M\in\mathbb{R}^{N\times N}. \tag{59}$$

i) When $|t-s|$ is small enough, we have

$$\mathcal{W}(t,s) = \mathcal{T}_{12}(t,s)^{-1}\circ\mathcal{P}_{1234}(t,s)\circ\mathcal{T}_{12}(s,s) \tag{60}$$

ii) If furthermore $A(t)\geq 0$, $D(t)\leq 0$, $\forall t$, and we choose $W_1,\ldots,W_4$ as in (55) and (57), then (60) is valid for all $t\geq s$. □

PROOF. Using (24) and (32) we find for small $|t-s|$:

$\mathcal{T}_{12}(W)(t,s) = G(\tilde{B},t,s)\mathcal{T}_{12}(W)(s,s)G^T(\tilde{B},t,s)$.

Using Lemma 2, we have

$$\mathcal{T}_{12}(W)(t,s) = G(\tilde{B},t,s)\mathbb{P}(\Delta)\cdot(\mathcal{T}_{12}(W)(s,s)\otimes U)\cdot\mathbb{P}^T(\Delta)\cdot G^T(\tilde{B},t,s).$$

Using Lemma 1 and (54),

$$\begin{aligned}\mathcal{T}_{12}(W)(t,s) &= \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s))\cdot G(\tilde{B},t,s)(\mathcal{T}_{12}(W(s,s)\otimes U)\times G^T(\tilde{B},t,s)\cdot\mathbb{P}^T(\mathcal{T}_{123}(W_4)(t,s))\\ &= \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s))\cdot[\mathcal{T}_{12}(W)(s,s)\otimes(G(\tilde{B},t,s)U\cdot G^T(\tilde{B},t,s))]\cdot\mathbb{P}^T(\mathcal{T}_{123}(W_4)(t,s))\\ &= \mathcal{P}_{1234}(\mathcal{T}_{12}(W)(s,s))\end{aligned}$$

the last equality following from $U = \mathbf{1}\cdot\mathbf{1}^T$, which proves i).

Under the hypotheses of ii), $W_1(s),\ldots,W_4(s)$ are all semidefinite positive matrices, so the corresponding solutions of the MRE exist for all $t\geq s$ (use the classical results associated with the optimal control problem giving rise to the MRE.) The previous computation is now valid for all $t\geq s$ and the domain of $\mathcal{W}(t,s)$ as a nonlinear operator on $\mathbb{R}^{N\times N}$ contains the set of symmetric semidefinite positive matrices. The theorem is proved. □

In the general case we shall use a fifth solution W_5 instead of X_1. Take W_4 and W_5 such that

$$\mathcal{T}_{123}(W_4)(s) = \Delta\ , \quad \mathcal{T}_{123}(W_5)(s) = U\ . \tag{61}$$

For example with the choice of W_1, W_2, and W_3 in (55), $W_4(s)$ is given by (57) and $W_5(s)$ is

$$W_5(s) = (I+U)^{-1} = I - \frac{1}{N+1} U. \tag{62}$$

The superposition law for the general case of the MRE (6) is given by the following theorem.

<u>THEOREM 5.</u> Let $W_1, \ldots, W_5$ be solutions of the MRE (6) satisfying (40) and (61) (take for example these solutions as in (55), (57) and (62) and set

$$\mathcal{P}_{12345}(t,s)(M) = \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s)) \cdot (M \otimes \mathcal{T}_{123}(W_5)(t,s)) \cdot \mathbb{P}^T(\mathcal{T}_{123}(W_4)(t,s))$$
$$\text{for} \quad M \in \mathbb{R}^{N\times N}\ . \tag{63}$$

When $|t-s|$ is small enough, we have

$$\mathcal{W}(t,s) = \mathcal{T}_{123}(t,s)^{-1} \circ \mathcal{P}_{12345}(t,s) \circ \mathcal{T}_{123}(s,s) \tag{64}$$

PROOF. Using (45) we find for small $|t-s|$:

$$\mathcal{T}_{123}(W)(t,s) = G(\tilde{B},t,s)\mathcal{T}_{123}(W)(s,s) \cdot G^{-1}(\tilde{B},t,s)$$

and using Lemma 2:

$$\begin{aligned}
\mathcal{T}_{123}(W)(t,s) &= G(\tilde{B},t,s)\mathbb{P}(\Delta) \cdot (\mathcal{T}_{123}(W)(s,s) \otimes U) \cdot \mathbb{P}^T(\Delta) G^{-1}(\tilde{B},t,s) \\
&= \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s)) \cdot G(\tilde{B},t,s) \cdot (\mathcal{T}_{123}(W)(s,s) \otimes U) \cdot G^{-1}(\tilde{B},t,s) \\
&\quad \cdot \mathbb{P}^T(\mathcal{T}_{123}(W_4)(t,s)) \\
&= \mathbb{P}(\mathcal{T}_{123}(W_4)(t,s)) \cdot [\mathcal{T}_{123}(W)(s,s) \otimes (G(\tilde{B},t,s) U \cdot G^{-1}(\tilde{B},t,s))] \\
&\quad \cdot \mathbb{P}^T(\mathcal{T}_{123}(W_4)(t,s)) \\
&= \mathcal{P}_{12345}(t,s)(\mathcal{T}_{123}(W)(s,s))
\end{aligned}$$

which follows from (61). This completes the proof. □

REMARKS. i) Formula (60) is a 4-representation of the general solution of the symplectic MRE in the sense that it utilizes four solutions of the MRE and one solution of a vector linear differential equation (58).

ii) Formula (64) is a 5-representation of the general solution of the MRE in the general case and a superposition law in the sense that $W(t)$ is only a function of the particular solutions $W_1(t),\ldots,W_5(t)$ and of the constant initial condition matrix $W(s)$.

5. CONCLUSIONS

The main results of this paper are summarized in Theorems 4 and 5 giving simple and explicit superposition formulas for the MRE (6). In the symplectic case the general solution is expressed in terms of 4 particular solutions of the MRE itself and one particular solution of the linear vector equation (58). In the general case of the MRE (6) precisely five particular solutions were used to express the general solution $W(t)$ (see (63),(64)).

In the 1-representation (17) additional information is used: the matrices H,G and S_0 are solutions of linear equations. Similarly, in the 2-representation (30), the linear evolution operator $L(\tilde{B},\tilde{C},t,s)$ (26) involves the matrices H and G obtained by solving linear equations. Finally, the representation (45) uses one solution of a linear equation, namely $G(\tilde{B},t,s)$ in the definition of $L(\tilde{B},-\tilde{B},t,s)$, in addition to the three solutions of the MRE.

The group theoretical origin of these formulas was elucidated in ref. [3]. The solution W(t) of the general MRE (6) was expressed in terms of the action of the group $SL(2N,\mathbb{R})$ on an initial condition constant matrix matrix W(s). The group element $g(t) \in SL(2N,R)$ can be completely reconstructed using 5 solutions, reconstructed up to an unknown matrix $g_2(t) \in SL(N,\mathbb{R})$ from 3 solutions, up to $g_1(t) \in SL(N,\mathbb{R})\otimes SL(N,\mathbb{R})$ from 2 solutions, and up to $g_0(t) \subset \left\{\begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22}\end{pmatrix}\right\}$, $G_{ik} \in \mathbb{R}^{N\times N}$, using 1 solution. The action of $g_0(t)$ is a linear inhomogeneous one, of $g_1(t)$ linear homogeneous and $g_2(t)$ linear and by conjugation. This explains why in each case the missing information for a representation formula can be obtained by solving the corresponding number of linear differential equations.

Let us now draw some conclusions from the formulas of this article, specially the 4-representation for the symplectic MRE and the 5-representation for the general MRE.

1. For nonsingular coefficients A,B,C,and D in the MRE (6) the solution space consists of meromorphic matrices: the matrix elements may have first order poles, the positions of which depend on the initial conditions. In other words, the MRE (6) has the "Painlevé property" [15]: the solutions have no moving critical points, i.e. branch points or essential singularities.

2. Each of the representations of this article can be used as a numerical method for solving the MRE. The 4- and 5-representations, i.e. the superposition formulas, are particularly efficient for calculating solutions that develop singularities (simple poles) for some finite time $t=t_0$.

Indeed, four, respectively five, continuous solutions can be chosen as input for the superposition formula and used as a "data bank" in the entire interval of interest, containing t_0. The solution can be calculated before and after $t=t_0$ and the exact value of t_0 can be pinpointed. The origin of the pole is clearly visible in (60) and (64): for a specific (discrete) value $t=t_0$ the inverse transformation $T_{12}(t,s)^{-1}$ or respectively $T_{123}(t,s)^{-1}$ does not exist. This method of continuing around singularities is to be compared with the addition formulas of Nelson et al[13,14].

3. In order to obtain simple and explicit superposition laws we have made use of a specific convenient set of particular solutions. This is not at all crucial: it was shown in ref. [3] that "fundamental sets of solutions" that can be used in a superposition formula form a dense set among all sets of five (or respectively four) solutions of the MRE.

An article on numerical applications [16] of the superposition laws for matrix Riccati equations is in preparation.

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